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On a new unified integral

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Abstract. In the present paper we derive a unified new integral whose integrand contains products of Fox H-function and a general class of polynomials having general arguments. A large number of integrals involving various simpler functions follow as special cases of this integral.

Keywords. Fox H-function; general class of polynomials; hypergeometric function.

1. Introduction

The H-function introduced by Fox [1], will be represented and defined in the following manner:

$$H_{p,q}^{m,n}[x] = H_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}x^{\xi}\,\mathrm{d}\xi. \quad (1.1)$$

For the nature of contour L in (1.1), the convergence, existence conditions and other details of the H-function, one can refer to [3].

The general class of polynomials introduced by Srivastava [4] is defined in the following manner:

$$S_V^U[x] = \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}A(V,K)}{K!}x^K, \quad V=0,1,2,\ldots, \quad (1.2)$$

where U is an arbitrary positive integer and coefficients $A(V,K),(V,K\geq 0)$ are arbitrary constants, real or complex.

2. Main result

$$\int_0^{\infty} x^{\lambda-1}[x+a+(x^2+2ax)^{1/2}]^{-\nu}H_{p,q}^{m,n}[y\{x+a+(x^2+2ax)^{1/2}\}^{-\mu}]$$
$$\times S_V^U[z\{x+a+(x^2+2ax)^{1/2}\}^{-\alpha}]\mathrm{d}x$$

$$= 2a^{-\nu}\left(\frac{1}{2}a\right)^{\lambda}\Gamma(2\lambda)\sum_{K=0}^{[V/U]}(-V)_{UK}A(V,K)\frac{(z/a^{\alpha})^K}{K!}H^{m,n+2}_{p+2,q+2}$$
$$\times\left[ya^{-\mu}\left|\begin{matrix}(-\nu-\alpha K,\mu),(1+\lambda-\nu-\alpha K,\mu),(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q),(1-\nu-\alpha K,\mu),(-\nu-\alpha K-\lambda,\mu)\end{matrix}\right.\right], \tag{2.1}$$

where

(i) $\mu > 0,\ \mathrm{Re}(\lambda,\nu,\alpha) > 0$,

(ii) $\mathrm{Re}(\lambda) - \mathrm{Re}(\nu) - \mu \min\limits_{1\le j\le m} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) < 0.$

Proof. To obtain the result (2.1), we first express Fox H-function involved in its left-hand side in terms of contour integral using eq. (1.1) and the general class of polynomials $S_V^U[x]$ in series form given by eq. (1.2). Interchanging the orders of integration and summation (which is permissible under the conditions stated with (2.1)) and evaluating the x-integral with the help of the result given below [2]:

$$\int_0^{\infty} x^{z-1}[x+a+(x^2+2ax)^{1/2}]^{-\nu}\mathrm{d}x$$
$$= 2\nu a^{-\nu}\left(\frac{1}{2}a\right)^{z}[\Gamma(1+\nu+z)]^{-1}\Gamma(2z)\Gamma(\nu-z), \quad 0<\mathrm{Re}(z)<\nu,$$

we easily arrive at the desired result (2.1).

3. Special case

If in the integral (2.1) we reduce $S_V^U[x]$ to unity and Fox H-function to Gauss hypergeometric function [3], we arrive at the following result after a little simplification:

$$\int_0^{\infty} x^{\lambda-1}[x+a+(x^2+2ax)^{1/2}]^{-\nu}$$
$$\times\, {}_2F_1(a,b;c;y(x+a+(x^2+2ax)^{1/2})^{-1})\mathrm{d}x$$
$$= 2^{1-\lambda}\nu\Gamma(2\lambda)a^{\lambda-\nu}\frac{\Gamma(\nu-\lambda)}{\Gamma(\nu+\lambda+1)}$$
$$\times\, {}_4F_3(a,b,\nu-\lambda,\nu+1;c,\nu,\nu+\lambda+1;y/a), \tag{3.1}$$

where

$$0<\mathrm{Re}(\lambda)<\mathrm{Re}(\nu), |y|<|a|.$$

The importance of the result given by (3.1) lies in the fact that it not only gives the value of the integral but also ‘augments’ the coefficients in the series in the integrand to give a ${}_4F_3$ series as the integrated series.

A number of other integrals involving functions that are special cases of Fox H-function [3] and/or the general class of polynomials [5] can also be obtained from (2.1) but we do not record them here.

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